Structural GARCH: The Volatility-Leverage Connection

Online Appendix

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Abstract

In this appendix, we: (i) show that total equity volatility is well approximated by the leverage multiplier in a general setup with stochastic volatility and jumps; (ii) provide an expression that can be evaluated to compute the contribution of jumps to equity volatility in the Bakshi, Cao, and Chen (1997) model; and (iii) show how our parameter estimates vary with our choice of the debt smoothing parameter and the maturities that we assign to different types of liabilities.

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1 How Well Does $LM_t \times \sigma_{A,t}$ Approximate Total Equity Volatility?

The purpose of this section is to show that the approximation in Equation (2) performs well in a fairly general option pricing setting. For completeness, recall that the asset return specification that we consider in the main text is given by:

$$\begin{aligned}\frac{dA_t}{A_t} &= [\mu_A(t) - \lambda\mu_J]dt + \sigma_{A,t}dB_A(t) + J_A dN_A(t) \\ d\sigma^2_{A,t} &= \mu_v(t, \sigma_{A,t})dt + \sigma_v(t, \sigma_{A,t})dB_v(t) \end{aligned} \tag{1}$$

J_a and $N_A(t)$ capture potential jumps in asset values. $\log(1 + J_A) \sim N\left(\log[1 + \mu_J] - \sigma_J^2/2, \sigma_J^2\right)$ and N_t is a Poisson counting process with intensity λ. As shown in Appendix A of the main text, when assets evolve as in Equation (1), the volatility of equity is given by:

$$vol_t\left(\frac{dE_t}{E_t}\right) = \sqrt{LM_t^2 \times \sigma^2_{A,t} + \frac{\nu_t^2\sigma_v^2(t, \sigma_{A.t})}{4E_t^2\sigma^2_{A,t}} + 2LM_t\sigma_{A,t}\frac{\nu_t}{E_t}\frac{\sigma_v(t, \sigma_{A,t})}{2\sigma_{A,t}}\rho_t + \mathcal{V}_A^J\left(J_A, N_A(t); A_t\right)} \tag{2}$$

where LM_t and ν_t derive from the underlying option pricing function, $f(\cdot)$. The term $\mathcal{V}_A^J\left(J_A, N_A(t); A_t\right)$ captures the volatility contribution coming from the jump component of asset returns. We provide a simple way to compute this term in Section 2.

Our goal is to validate the following approximation of equity volatility:

$$\begin{aligned}\frac{dE_t}{E_t} &\approx LM_t\sigma_{A,t}dB_A(t) \\ vol_t\left(\frac{dE_t}{E_t}\right) &\approx LM_t \times \sigma_{A,t}\end{aligned} \tag{3}$$

The main intuition behind this approximation is as follows: as the time to maturity of the equity option gets larger, the contribution of the volatility of volatility to equity returns/volatility is minimal. This is because volatility is mean reverts much faster than the maturity of debt, and thus the total volatility over the life of the option is effectively constant. Ultimately, this means that volatility of asset volatility matters less for equity return volatility. A similar logic applies when asset returns experience jumps, so long as the jumps occur with a small enough probability and the jump size is itself not too volatile.

To assess the accuracy of the approximation in Equation (3), we analyze the Bakshi, Cao, and Chen (1997) (BCC) model. We choose this model because it includes both stochastic volatility and jumps, so we think it represents a fairly general option pricing setting. In the BCC model, risk-neutral asset return dynamics are described as follows:

$$\begin{aligned}\frac{dA_t}{A_t} &= \left[r_{ft} - \lambda_J \mu_J\right] dt + \sigma_{A,t} dB_A(t) + J(t) dN(t) \\ d\sigma^2_{A,t} &= \left[\theta_v - \kappa_v \sigma^2_{A,t}\right] dt + \sigma_{BCC,v}\sigma_{A,t} dB_v(t)\end{aligned}$$

where $N(t)$ is a Poisson counting process with constant intensity λ_J. $J(t)$ is the percentage jump size and is i.i.d lognormal with unconditional mean μ_J and standard deviation σ_J. The parameter θ_v dictates the long-run average of risk-neutral asset volatility and κ_v determines the speed of mean reversion. Finally, the correlation between $dB_v(t)$ and $dB_A(t)$ is given by ρ.

The first thing we must do is map the asset return specification in the BCC model to the asset return specification in Equation (1). The relevant mapping for our purposes is:

$$\sigma_v = \sigma_{BCC,v}\sigma_{A,t}$$

With this in hand, we can use Equation (2) to express the total volatility of equity returns in terms of the underling parameters in the BCC model:

$$vol_t\left(\frac{dE_t}{E_t}\right) = \sqrt{LM_t^2 \times \sigma^2_{A,t} + \frac{\nu_t^2 \sigma^2_{BCC,v}}{4E_t^2} + \frac{LM_t \sigma_{A,t} \nu_t \sigma_{BCC,v} \rho_t}{E_t} + \mathcal{V}^J_A\left(J_A, N_A(t); A_t\right)} \quad (4)$$

Armed with Equation (4), we are now in a position to assess the approximation for equity volatility given in Equation (3), at least inside of the BCC model. To do so, we use the same parameterization of the BCC model that we use in the main text, which we repeat here for convenience. We set $\lambda_J = 0.4, \mu_J = -0.1, \sigma_J = 0.15$. This corresponds to 0.4 jumps a year, each with an average jump size of -10% and volatility of 15%. Additionally, we set $\kappa_v = 4.08$ and $\theta_v = 0.013$. This means that the half-life of volatility is about 66 days (roughly what is typically found in the literature) and the unconditional volatility of asset returns is 17.5%. We also set $\sigma_v = 17.5\%$ and $\rho = -0.7$. When computing the leverage multiplier, we set the current spot volatility equal to its long run average, so $\sigma_{A,0} = 17.5\%$.

In addition, we consider time to maturities of $\tau = 2, 5, 10$. This allows us to get a sense of how well the approximation works for different debt maturities. In all cases, we set the annualized risk free rate $r = 3\%$. For each maturity, we vary the face value of debt, and compute the ratio of Equation (3) to Equation (4). We define this ratio as Ψ, and it represents the proportion of total volatility that comes from the simple leverage multiplier term that we use as a workhorse throughout the paper. Keep in mind that Ψ is computed for each value of debt, D. We can also use the BCC model to compute the implied equity value for each value of D. Finally, in Figure 1 we plot each Ψ against its corresponding implied debt-to-equity ratio.

The main takeaway from Figure 1 is that Equation (3) is a pretty good approximation of total equity volatility. Regardless of debt maturity, the portion of total volatility coming from $LM_t \times \sigma_{A,t}$ is very high, ranging from 85% to 110%. For reasonable amounts of leverage (e.g. 5-15), this proportion is anywhere from 90-105%. Obviously these numbers depend heavily on the calibration we have used; nevertheless, it seems reasonable to at least conclude that the first order determinant of total equity volatility comes from how the leverage multiplier amplifies asset volatility.

2 Total Volatility of Equity in Stochastic Volatility with Jump Environment

We need to compute the variance of the following term:

$$\mathcal{V}_A^J\left(J_A, N_A(t); E_t\right) \equiv var_t\left(J_E dN_A(t)\right)$$

where J_E is the percentage change in equity induced by jumps. Clearly, this is a function of the percentage change in assets induced by jumps. Explicitly, this is:

$$\begin{aligned} J_E &= \frac{E_t^J - E_t}{E_t} \\ &= \frac{f(A_t(1+J_A))}{f(A_t)} - 1 \end{aligned}$$

where $f(\cdot)$ is the call option pricing function (we suppress all but the first argument for notional convenience). The definition of variance implies that:

$$\begin{aligned}
\mathcal{V}_A^J\left(J_A, N_A(t); E_t\right) &= \mathbb{E}\left[\left(J_E dN_A(t)\right)^2\right] - \mathbb{E}\left[J_E dN_A(t)\right]^2 \\
&= \mathbb{E}\left[J_E^2 dN_A(t)^2\right] - \mathbb{E}\left[J_E\right]^2 \mathbb{E}\left[dN_A(t)\right]^2 \\
&= \mathbb{E}\left[J_E^2\right] \mathbb{E}\left[dN_A(t)^2\right] - \mathbb{E}\left[J_E\right]^2 \lambda^2 dt^2 \\
&= \mathbb{E}\left[J_E^2\right] \lambda dt
\end{aligned}$$

Here, the second and third lines use the independence of the asset jump, J_A (and hence J_E), and the Poisson process $dN_A(t)$. Additionally, the third and fourth lines use the standard variance definition for a Poisson process at small time intervals.[1]

All that remains is to compute the following term:

$$\begin{aligned}
\psi(A_t, J_A) &\equiv \mathbb{E}\left[J_E^2\right] \\
&= \mathbb{E}\left[\left(\frac{f(A_t(1+J_A))}{f(A_t)} - 1\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{f(A_t e^Y)}{f(A_t)} - 1\right)^2\right] \\
&= \int_{-\infty}^{\infty} \left(\frac{f(A_t e^y)}{f(A_t)} - 1\right)^2 h_Y(y) dy
\end{aligned}$$

where $h_Y(y)$ is the pdf of a normal random variable with mean $\log\left[1+\mu_J\right] - \sigma_J^2/2$ and variance σ_J^2. In practice, this is easily computed numerically. We have further checked that this derivation is correct by computing the variance contribution of jumps (to equity volatility) when there is no debt. This must correspond to the case where assets equal equity, in which case there is a closed form solution for the component of variance coming from jumps (Bakshi, Cao, and Chen (1997)). We can then compare the closed form solution to our numerical

[1] i.e.

$$\begin{aligned}
var_t\left(dN_A(t)\right) &= \mathbb{E}\left[dN_A(t)^2\right] - \mathbb{E}\left[dN_A(t)\right]^2 \\
&\Leftrightarrow \\
\mathbb{E}\left[dN_A(t)^2\right] &= var_t\left(dN_A(t)\right) + \mathbb{E}\left[dN_A(t)\right]^2 \\
&= \lambda dt + \lambda^2 dt^2 \\
&= \lambda dt
\end{aligned}$$

computation above.

3 Robustness

In this section, we explore the robustness of our parameter estimates to two different assumptions: (i) the smoothing parameter that we use to exponentially smooth debt; and (ii) the maturities that we assign to each liability type. All robustness checks are found in Table 1.

3.1 Smoothing Parameter for Debt

As a reminder, we define the face value of debt D_t as the sum of insurance reserves, deposits, short term debt, long term debt, and other liabilities. We observe debt values at the end of each quarter. For each day in a given quarter, we use the debt value reported at the end of the last quarter. This naturally creates a very choppy debt series. To minimize the impact that this has on the estimation, we smooth the daily book value of debt using an exponential average:

$$D_t = \eta \widehat{D}_t + (1-\eta) D_{t-1}, \qquad D_0 = \widehat{D}_0$$

In our baseline model, we set $\eta = 0.01$. We chose this value because it implies a half-life of approximately 70 days in terms of the weights of the exponential average. This seems reasonable for quarterly data. Table 1 presents the median parameter estimates when we use $\eta = 0.05$. For convenience, we report our baseline median parameter estimates in the table as well (this is the first set of parameter estimates in the table). When using $\eta = 0.05$, the Structural GARCH parameter estimates are basically the same as when $\eta = 0.01$. In addition, the number of firms with ϕ that is statistically different from zero is also insensitive to the choice of η.

3.2 Debt Maturity

Recall that to compute the debt maturity for every firm at each point in time, we first assign a maturity to each type of liability on the firm's balance sheets. For all firms, we set the maturity of each liability as follows: insurance reserves are 30 years, deposits are 1 year, short term debt is 2 years, long term debt is 8 years, and other liabilities are 3 years. These maturities reflect our best guess of the "typical" maturity of each type of liability. We now check how sensitive our parameter estimates are to different assumptions about the maturity type of each liability.

Deposits

Deposits are by definition very short term debt. However, one might argue that deposit insurance means banks and markets view deposits as having very little rollover risk. Under this view, deposits are actually much longer duration liabilities. In lieu of this possibility, we re-estimate our baseline model but set the maturity of deposits to 10 years instead of 1 year. As Table 1 shows, the median GJR parameter estimates for asset returns are largely unaffected by changing the assumed maturity of deposits. The most sensitive parameter is clearly ϕ, which moves from a baseline value of 0.68 to 0.89 with longer duration deposits. Intuitively, increasing the time to maturity of debt mechanically lowers the leverage multiplier because there is more time for assets to expire "in the money". Because the data calls for a larger leverage multiplier, the estimated ϕ increases in order to offset this effect. Importantly, the number of firms with a statistically significant ϕ is unchanged from this exercise.

Other Liabilities

The "other liabilities" item on firms' balance sheets is also difficult to pin down in terms of maturity. We (somewhat arbitrarily) set other liabilities to have a maturity of 3 years. From Table 1, it is clear that changing the maturity of other liabilities from 3 to 7 years has basically no impact on the baseline parameter estimates. This is most likely because other liabilities are such a small portion of most firms' balance sheets.

Insurance Reserves

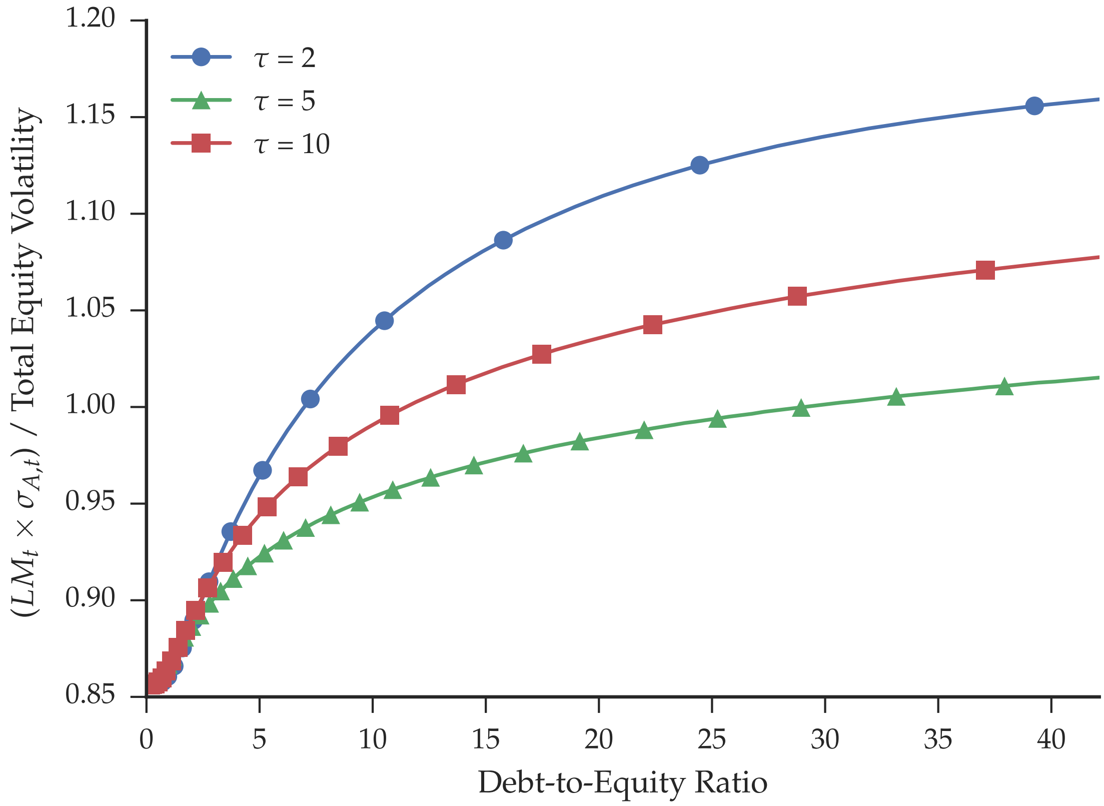
While it is clear that insurance reserves should have a fairly long maturity, it is not obvious precisely what number this should be. Our last perturbation is to change the maturity of insurance reserves from 30 years to 20 years. The last set of parameter estimates in Table 1 indicate that the moving insurance reserves to a 20 year maturity does not impact the parameter estimates in a material way. This is likely for two reasons. The first is that insurance reserves are not a huge liability for many of the firms in our sample. The second is that the leverage multiplier becomes less sensitive to changes in the time to maturity for very large maturities.

Details on Maturity of Long-Term Debt

In the baseline estimation of the model, we define the maturity of a firm's total liability structure to be a weighted-average of the maturities of its deposits, short term debt, long term debt, and insurance reservers. In order to operationalize this definition, we use a "best-guess" of the maturities for each of these debt instruments. To inform our best guess of long term debt maturity, we downloaded a snapshot of the outstanding bonds for the firms in our sample. This data is available from Bloomberg and we used data as of 9/6/2016.[2] Bloomberg reports the average debt maturity across the different vintages of each firm's bonds, which we call $\overline{\tau}_{i,LTD}$. Across our subset the average $\overline{\tau}_{i,LTD}$ is 8.78 years and the median is 6.90 years. We split the difference between these two numbers and set long term debt to have a (rounded) maturity of 8.

[2] 7 of the firms did not have data available on Bloomberg.

Figure 1: How Well Does $LM_t \times \sigma_{A,t}$ Approximate Total Equity Volatility?



Notes: This figure visualizes the accuracy of using $LM_t \times \sigma_{A,t}$ in approximating total equity volatility. To assess this accuracy, we compute both $LM_t \times \sigma_{A,t}$ and total equity volatility in the Bakshi, Cao, and Chen (1997) (BCC) model. In the BCC model, asset returns have stochastic volatility and experience jumps. For a given time to maturity τ, we vary the face value of debt in the BCC model and compute the ratio of $LM_t \times \sigma_{A,t}$ to total equity volatility. Total equity volatility is given by Equation (4). The figure plots this ratio (for a given debt and τ) against the implied debt-to-equity ratio (for the same debt and τ). We consider three maturities of $\tau = 2, 5, 10$. For all data points, the interest rate $r = 0.03$, the time to maturity $\tau = 2$, and the total volatility of assets is set to 17.5%. Furthermore, we calibrate the Bakshi, Cao, and Chen (1997) model as follows: the speed of volatility mean reversion $\kappa = 2.77$; the correlation between volatility shocks and asset return shocks $\rho = -0.7$; long-run average asset variance $\theta = 0.013$; volatility of asset variance $\sigma_v = 17.5\%$; jump intensity $\lambda_J = 0.4$; average jump size $\mu_J = -0.1$; and average jump volatility $\sigma_J = 0.15$. All parameters are stated in risk-neutral space.

Table 1: Parameter Sensitivity Analysis

τ_D	τ_S	τ_L	τ_I	τ_O	η	$\omega \times 10^7$	α	γ	β	ϕ	% with $\lvert t(\phi)\rvert \geq 1.64$
1	2	8	30	3	0.01	7.30	0.041	0.071	0.914	0.680	
						(1.12)	(2.91)	(3.01)	(71.24)	(2.36)	60.4
1	2	8	30	3	0.05	7.56	0.042	0.071	0.914	0.696	
						(1.22)	(2.88)	(2.82)	(69.85)	(2.35)	61.5
10	2	8	30	3	0.01	8.01	0.040	0.072	0.914	0.890	
						(1.17)	(2.92)	(3.06)	(71.30)	(2.77)	62.6
1	2	8	30	7	0.01	7.10	0.042	0.071	0.914	0.684	
						(1.16)	(2.96)	(3.05)	(71.47)	(2.24)	59.3
1	2	8	20	3	0.01	6.86	0.041	0.072	0.913	0.709	
						(1.12)	(2.91)	(3.01)	(71.24)	(2.70)	64.8

Notes: This table shows how sensitive the Structural GARCH parameters are to our choice of debt maturities for various liabilities, as well as the smoothing parameter that we use when exponentially smoothing the total debt for each firm. τ_i is the maturity (years) of liability type i, where $i = D$ is deposits, S is short-term debt, L is long-term debt, I is insurance reserves, and O is other liabilities. η is the smoothing parameter we use when exponentially smoothing the total firm debt series, where total firm debt is the sum of all of the liability types. The reported Structural GARCH parameters are the median point estimates across the 91 firms in our sample. Median Bollerslev-Wooldridge robust t-statistics are listed below each parameter value in parentheses. % with $|t(\phi)| \geq 1.64$ indicates the percent of firms with a ϕ that is statistically different from zero at a 10 percent confidence level. The first set of parameters listed is the baseline configuration that we use for all estimation in the main text.

References

Bakshi, G., C. Cao, and Z. Chen (1997). Empirical performance of alternative option pricing models. *The Journal of finance 52*(5), 2003–2049.